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Another Critical Exponent Inequality for Percolation: $\beta \geqslant 2/\delta$

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The inequality in the title is derived for standard site percolation in any dimension, assuming only that the percolation density vanishes at the critical point. The proof, based on a lattice animal expansion, is fairly simple and is applicable to rather general (site or bond, short- or long-range) independent percolation models.

KEY WORDS: Percolation; critical exponents; exponent inequalities; rigorous results; lattice animals.

1. PREFACE

The main body of this paper concerns a new critical exponent inequality for percolation, which was presented at the 1986 Třeboň symposium. References for the other topics of my Třeboň talks are as follows. A brief survey of rigorous percolation theory may be found in Ref. 13 (other recent surveys include Refs. 1 and 11); included as an appendix of Ref. 13 is a previously unpublished 1981 preprint on the relation between Burgers' equation and the phase transition in Ising ferromagnets. Results concerning the phase transition in one-dimensional $1/r^2$ percolation, Ising, and Potts models appear in Refs. 3, 5, 7, and 16. In addition, Ref. 3 contains general results concerning the Fortuin–Kastelyn representation of Ising and Potts models.[10,12] Applications of the Fortuin–Kastelyn representation to dilute Ising and Potts models will be the topic of a separate paper.[4]

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2. INTRODUCTION

In this paper, I derive a new inequality,

$$\beta \geqslant 2/\delta \tag{1}$$

relating the critical behavior of the cluster size distribution $[P_n(p_c) \sim n^{-1-1/\delta}$ as $n \to \infty]$ to that of the percolation density $[P_\infty(p) \sim (p-p_c)^\beta$ as $p \downarrow p_c]$. The proof is a simple one based on lattice animal expansions and the inequality is valid, assuming only that $0 < p_c < 1$ and that $P_\infty(p_c) = 0$, for essentially any (site or bond, short- or long-range) independent, translation-invariant percolation model in any dimension d. For simplicity, however, attention here will be restricted to standard site percolation; the interested reader can find in Ref. 6 a presentation of the type of lattice animal expansion needed to treat general bond percolation models. Rigorous exponent inequalities other than those discussed here may be found in review papers[1,11,13] and the references given there.

The inequality (1) improves the previous results[2] that

$$\beta \geqslant 1/(\delta - 1) \tag{2}$$

and that (see also Ref. 9) $\delta \geqslant 2$, since $\beta \leqslant 1$.[8] It should be noted though that the proof of inequality (1) yields nothing when $P_\infty(p_c) > 0$, but that of (2) implies $\beta \geqslant 1/\delta$. In certain long-range one-dimensional models, $P_\infty(p_c) > 0$.[7]

While both inequalities (1) and (2) saturate for the percolation mean field values ($\beta = 1$, $\delta = 2$), the second is actually a "universal" mean field inequality, which is also valid for Ising models.[13] Apparently, the first inequality is stronger because it is less universal. The rigorous inequality (1), when compared to the nonrigorous values of the percolation exponents in low dimensions (see, e.g., Ref. 18), performs fairly well; it misses saturation by less than 25% for $d = 2$ and less than 10% for $d = 3$.

3. THE MAIN RESULT

We consider nearest neighbor site percolation on the d-dimensional hypercubic lattice $\mathbb{Z}^d$ with site occupation probability p. We let $P_n(p)$ denote the probability that the cluster of the origin contains exactly n occupied sites ($n = 0, 1,..., \infty$). The critical point p_c is the largest p with $P_\infty(p) = 0$, i.e.,

$$p_c = \sup\{p: P_\infty(p) = 0\} \tag{3}$$

We also define, for $h > 0$,

$$M_c(h) = 1 - \sum_{n=0}^{\infty} P_n(p_c)\, e^{-nh} \tag{4}$$

Note that δ may be defined by

$$M_c(h) - P_\infty(p_c) \sim h^{1/\delta} \qquad \text{as} \quad h \downarrow 0$$

The inequality (1) is then an immediate consequence of the next proposition, providing $P_\infty(p_c) = 0$. The vanishing of P_∞ at the critical point, although expected to be valid for all d, has only been rigorously proved for $d = 2$[17] [but see Ref. 6, Prop. 1.3ff., and Ref. 14 for sufficient conditions for $P_\infty(p_c) = 0$].

Proposition 1. For any $d > 1$, there is some finite, positive constant K so that

$$[P_\infty(p_c + \varepsilon)]/[M_c(K\varepsilon^2)] \quad \text{is bounded as} \quad \varepsilon \downarrow 0 \tag{5}$$

Proof. We use the standard representation

$$P_n(p) = \sum_l P_{nl}(p) \equiv \sum_l a_{nl}\, p^n (1-p)^l \tag{6}$$

where a_{nl} denotes the number of lattice animals with n occupied sites and l vacant boundary sites. From the definition (6) of P_{nl}, it follows that

$$P_{nl}(p \pm \varepsilon) = (1 \pm \varepsilon/p)^n [1 \mp \varepsilon/(1-p)]^l\, P_{nl}(p) \tag{7}$$

Our basic string of estimates is then

$$\begin{aligned}
&[1 - P_\infty(p-\varepsilon)][1 - P_\infty(p+\varepsilon)] \\
&\quad = \left[\sum_{n,l} P_{nl}(p-\varepsilon)\right]\left[\sum_{n,l} P_{nl}(p+\varepsilon)\right] \\
&\quad = \left(\sum_{n,l} \{(1-\varepsilon/p)^{n/2}[1+\varepsilon/(1-p)]^{l/2}\}^2 P_{nl}(p)\right) \\
&\qquad \times \left(\sum_{n,l} \{(1+\varepsilon/p)^{n/2}[1-\varepsilon/(1-p)]^{l/2}\}^2 P_{nl}(p)\right) \\
&\quad \geqslant \left\{\sum_{n,l} (1-\varepsilon^2/p^2)^{n/2}[1-\varepsilon^2/(1-p)^2]^{l/2} P_{nl}(p)\right\}^2 \\
&\quad \geqslant [1-\varepsilon^2/(1-p)^2] \left\{\sum_{n,l} \exp[-K(p)\,\varepsilon^2 n]\, P_{nl}(p)\right\}^2
\end{aligned} \tag{8}$$

where the sums are all restricted to $n < \infty$. The first inequality of (8) is just the Cauchy–Schwarz inequality for doubly indexed sequences. The second inequality, which is only valid for small ε, follows from the simple fact that a_{nl} (and hence P_{nl}) vanishes unless $l \leqslant (2d-1)n+1$ (for $n = 0, 1, 2,...$), combined with the estimate that $1-u > e^{-2u}$ for small, positive u. The quantity $K(p)$ is given by

$$K(p) = p^{-2} + (2d-1)(1-p)^{-2}$$

We set $p = p_c$ in (8) to obtain

$$\begin{aligned} P_\infty(p_c+\varepsilon) &\leqslant 1 - [1-\varepsilon^2/(1-p_c)^2][1-M_c(K\varepsilon^2)]^2 \\ &\leqslant 2M_c(K\varepsilon^2) + \varepsilon^2/(1-p_c)^2 \end{aligned} \tag{9}$$

where $K = K(p_c)$. Since $h/M_c(h)$ is easily seen to stay bounded as $h \downarrow 0$ (it actually tends to zero, but we will not use this), the proof of (5) is complete.

Remark. The exponent γ is determined by

$$\chi(p) = \sum_{n<\infty} nP_n(p)$$

according to $\chi(p) \sim (p_c - p)^{-\gamma}$ as $p \uparrow p_c$, and γ' is defined analogously as $p \downarrow p_c$. It is possible to modify the arguments used above to obtain a different (and perhaps simpler) derivation of the inequalities,

$$\gamma, \gamma' \geqslant 2(1-1/\delta) \tag{10}$$

than the one given in Ref. 14. This new derivation is closely related to the one used in the last remark of Ref. 15, which yielded $(\gamma+\gamma')/2 \geqslant 2(1-1/\delta)$. The required modification of (8) involves the insertion of an extra factor of n in the summands and the use of Hölder's inequality instead of the Cauchy–Schwarz inequality. One finds that for $0 < a < 1$,

$$\begin{aligned} &[\chi(p_c - (1-a)\,\varepsilon)]^a[\chi(p_c + a\varepsilon)]^{1-a} \\ &\qquad \geqslant \sum_{n<\infty} ne^{-nK\varepsilon^2} P_n(p_c) = \frac{dM_c}{dh}(K\varepsilon^2) \end{aligned} \tag{11}$$

for small ε, where this K depends on both p_c and a. Thus, if we take δ as defined by $dM_c/dh \sim h^{1/\delta - 1}$, it follows that $a\gamma + (1-a)\gamma' \geqslant 2(1-1/\delta)$ for any a in (0, 1), which yields (10).

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